Zoltán Füredi

On a theorem of Erdős and Simonovits on graphs not containing the cube

Abstract: The cube Q is the usual 8-vertex graph with 12 edges. Here we give a new proof for a theorem of Erdős and Simonovits concerning the Turán number of the cube. Namely, it is shown that $e(G) \le n^{8/5} + (2n)^{3/2}$ holds for any n-vertex cube-free graph G.

Our aim is to give a self-contained exposition. We also point out the best known results and supply bipartite versions.

Keywords: Turán graph problem, bipartite extremal graphs, cube graph

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Zoltán Füredi: Rényi Institute of Mathematics of the Hungarian Academy of Sciences, Budapest, P.O. Box 127, Hungary-1364, E-mail: z-furedi@illinois.edu, furedi.zoltan@renyi.mta.hu

1 History of Turán type problems

As usual, we write $|G|$, $e(G)$, $\deg_G(x)$ for the number of vertices, number of edges, and the degree of a vertex x of a graph G. Denote by $N_G(x)$ (or just $N(x)$) the neighborhood of x, note that $x \notin N(x)$. Let K_n and $K_{a,b}$ denote the complete graph on n vertices and the complete bipartite graph with bipartition classes of sizes a and b. $K(A, B)$ denotes the complete bipartite graph with partite sets A and B ($A \cap B = \emptyset$).

A graph G not containing H as a (not necessarily induced) subgraph is called *H-free*. Let us denote by $\mathbf{ex}(n, H)$ the *Turán number* for H, i.e. the maximum number of edges of an H-free graph on n vertices. More generally, let $\mathbf{ex}(G, H)$ be the maximum number of edges in an H-free subgraph of G. Then $\mathbf{ex}(n, H) = \mathbf{ex}(K_n, H)$. We also use the notation $\mathbf{ex}(a, b, H)$ for $\mathbf{ex}(K_{a,b}, H)$ and call it the bipartite version of Turán number. Also, if $F \subset H$ then $\mathbf{ex}(n, F) \le \mathbf{ex}(n, H)$.

Turán [29] determined $\mathbf{ex}(n, K_{p+1})$. The extremal graph is the almost equipartite complete graph of p classes. He also proposed the general question, $\mathbf{ex}(n, H)$, in particular the determination of the Turán number of the graphs obtained from the platonic polyhedrons, the cube $Q = Q_8$ (it is an 8-vertex 3-regular graph), the octahedron O_6 (six vertices, 12 edges), the icosahedron I_{12} (12 vertices, 5-regular) and for the dodecahedron D_{20} (20 vertices, 30 edges). Erdős and Simonovits [12] gave an implicit formula for $\mathbf{ex}(n, O_6)$ (they reduced it to $\mathbf{ex}(n, C_4)$) and Simonovits solved exactly $\mathbf{ex}(n, D_{20})$ in [26] and $\mathbf{ex}(n, I_{12})$ in [27] (for $n > n_0$).

In fact, Turán's real aim was not only these particular graphs but to discover a general theory. His questions, and the answers above, indeed lead to an asymptotic (the

Erdős–Simonovits theorem [10]) and to the Simonovits stability theorem concerning the extremal graphs [25] in the case when the sample graph H has chromatic number at least three. For a survey and explanation see Simonovits [28] or the monograph of Bollobás [5].

However, the bipartite case is different, see the recent survey [16]. Even the extremal problem of the cube graph Q, which was one of Turán's [30] originally posed problems, is still unsolved. Our aim here is to give a gentle introduction to this topic. We survey the results and methods concerning $\mathbf{ex}(n, Q)$, give new or at least streamlined proofs. We only use basic ideas of multilinear optimization (Lagrangian, convexity, etc.) and in most cases just high school algebra. We also consider the case of bipartite host graph, i.e., $\mathbf{ex}(a, b, Q)$.

2 Walks

Let $W_3 = W_3(G)$ denote the number of walks in G of length 3, i.e., the number of sequences of the form $x_0x_1x_2x_3$ where $x_{i-1}x_i$ is an edge of G (for $i = 1, 2, 3$). Note that, e.g., $xyxy$ is a walk (if $xy \in E(G)$) and it differs from $yxyx$. A d-regular graph has exactly nd^3 3-walks.

Theorem 1. *For every n-vertex graph G for the number of 3-walks one has*

$$W_3 \geq n\left(\frac{1}{n}\sum_{x\in V} \deg(x)^{3/2}\right)^2. \tag{1}$$

The r-order *power mean* of the nonnegative sequence $a_1, \dots, a_m$ is $M_r(\mathbf{a}) := \left(\frac{1}{m}\sum a_i^r\right)^{1/r}$. Then for $1 \leq r \leq s \leq \infty$ one has

$$a_{\text{ave}} := M_1(\mathbf{a}) \leq M_r(\mathbf{a}) \leq M_s(\mathbf{a}) \leq M_\infty(\mathbf{a}) := \max_i |a_i|. \tag{2}$$

We will frequently use it in the equivalent form

$$\sum_{1\leq i\leq m} a_i^r \leq \left(\sum a_i^s\right)^{r/s} m^{1-(r/s)}. \tag{3}$$

This is just a special case of the Hölder inequality, i.e., for any two nonnegative vectors $\mathbf{x}, \mathbf{y} \in R^m$ and for reals $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ one has

$$\sum_i x_i y_i \leq \left(\sum_i x_i^p\right)^{1/p} \left(\sum_i y_i^q\right)^{1/q}.$$

We get (3) by substituting here $\mathbf{x} = (a_i^r)_{1\leq i\leq m}$, $\mathbf{y} = (1, 1, \dots, 1)$, $1/p = r/s$ and $1/q = 1 - (r/s)$.

Proof of Theorem 1. Considering the middle edge of the 3-walks one obtains that

$$W_3 = \sum_{x\in V}\sum_{y\in N(x)} \deg(x)\deg(y).$$

Here we have $2e = nd_{\text{ave}}$ terms. We use for this sum the $2e$-dimensional Chauchy–Schwartz inequality

$$\Big(\sum_i a_i^2\Big)\Big(\sum_i b_i^2\Big) \geq \Big(\sum_i a_i b_i\Big)^2$$

valid for any two vectors $\mathbf{a}, \mathbf{b} \in R^m$. Our aim is to separate the variables in the products $\deg(x)\deg(y)$ so we take $\mathbf{a} = \{\sqrt{\deg(x)\deg(y)}\}_{x\in V,\, y\in N(x)}$ and $\mathbf{b} = \{\sqrt{1/\deg(x)}\}_{x\in V,\, y\in N(x)}$. One obtains that

$$\begin{aligned} W_3\, n &= \Big(\sum_{x\in V}\sum_{y\in N(x)} \deg(x)\deg(y)\Big)\Big(\sum_{x\in V}\sum_{y\in N(x)} \frac{1}{\deg(x)}\Big) = \Big(\sum_i a_i^2\Big)\Big(\sum_i b_i^2\Big) \\ &\geq \Big(\sum_i a_i b_i\Big)^2 = \Big(\sum_{x\in V}\sum_{y\in N(x)} \frac{\sqrt{\deg(x)\deg(y)}}{\sqrt{\deg(x)}}\Big)^2 = \Big(\sum_{x\in V}\sum_{y\in N(x)} \sqrt{\deg(y)}\Big)^2 \\ &= \Big(\sum_{y\in V} \deg(y)^{3/2}\Big)^2. \end{aligned}$$

□

Historical remarks. One can rewrite Theorem 1 as

$$W_3 \geq nM_{3/2}(\mathbf{d})^3. \tag{4}$$

Then the power mean inequality (2) with $(r, s) = (1, 3/2)$ gives that

$$W_3 \geq n(d_{\text{ave}})^3 = 8e^3/n^2. \tag{5}$$

This inequality $W_3 \geq n(d_{\text{ave}})^3$ is due to Mulholland and C. A. B. Smith [20] and was generalized by Attkinson, Watterson and Moran [3] for W_k for every $k \geq 3$ in a form of a matrix inequality. Then it was further generalized by Blakley and Roy [4] for all non-negative symmetric matrices. As far as the author knows the obvious consequence of their works, $W_k \geq n(d_{\text{ave}})^k$, was first explicitly stated in a paper of Erdős and Simonovits [13]. For the interested reader we supply a direct proof for (5) using only high school algebra in the Appendix (Section 8).

Theorem 1 is not really new. It is an easy consequence (of a special case) of a result of Jagger, Šťovíček, and Thomason [18], who while working on a conjecture of Sidorenko [24] showed the inequality $\sum_x w(x)^{1/2} \geq \sum_x \deg(x)^{3/2}$ where $w(x)$ is the number of 3-walks whose second vertex is x.

The exponent $3/2$ is the best possible. Consider a complete bipartite graph $K_{a,b}$, we have $W_3 = 2a^2b^2$. Then $W_3/nM_p(\mathbf{d})^3 \to 0$ for any fixed $p > 3/2$ whenever $b/a \to \infty$.

For $K_{a,b}$ we have $W_3 = 2a^2b^2$, while the right-hand side of (1) is $a^2b^2\frac{(\sqrt{a}+\sqrt{b})^2}{a+b}$ which is between a^2b^2 and $2a^2b^2$. Using this observation one can show the following: Suppose that $d_1, \dots, d_n$ is the degree sequence of a graph G. There is a graph H with degree sequence d_i' for which $d_i' \geq d_i$ and $W_3(H) \leq 4n(\sum(d_i')^{3/2}/n)^2 = 4nM_{3/2}(\mathbf{d}')^3$.

3 3-paths in bipartite graphs

Let $P_3 = P_3(G)$ denote the number of 3-paths of G. We have $2P_3 \le W_3$. Using the method of the previous section we show the following lower bound for P_3.

Theorem 2. *Let $G(A, B)$ be a bipartite graph with e edges and with color classes A and B, $|A| = a$, $|B| = b$. Suppose that every vertex has degree at least 2. Then for the number of 3-paths one has*

$$P_3 \ge \frac{e(e-a)(e-b)}{ab}. \tag{6}$$

Proof. Considering the middle edge of the 3-paths one obtains that

$$P_3 = \sum_{x\in A}\sum_{y\in N(x)} (\deg(x)-1)(\deg(y)-1).$$

Here we have e terms. One obtains that

$$\begin{aligned} a\,P_3 &= a\sum_{x\in A}\sum_{y\in N(x)} (\deg(x)-1)(\deg(y)-1)\\ &= a\sum_{x\in A}\sum_{y\in N(x)} -(\deg(y)-1) + a\sum_{x\in A}\sum_{y\in N(x)} \deg(x)(\deg(y)-1)\\ &= -a\sum_{y\in B}\deg(y)(\deg(y)-1) + \Big(\sum_{x\in A}\sum_{y\in N(x)}\frac{1}{\deg(x)}\Big)\Big(\sum_{x\in A}\sum_{y\in N(x)}\deg(x)(\deg(y)-1)\Big). \end{aligned}$$

Here the second term is at least

$$\begin{aligned} &\ge \Big(\sum_{x\in A}\sum_{y\in N(x)}\frac{\sqrt{\deg(x)(\deg(y)-1)}}{\sqrt{\deg(x)}}\Big)^2 = \Big(\sum_{x\in A}\sum_{y\in N(x)}\sqrt{\deg(y)-1}\Big)^2\\ &= \Big(\sum_{y\in B}\deg(y)\sqrt{\deg(y)-1}\Big)^2. \end{aligned}$$

Let $F(y_1, y_2, \dots, y_b)$ be a real function defined as

$$-a\sum_{1\le i\le b}(y_i^2 - y_i) + \Big(\sum_i y_i\sqrt{y_i-1}\Big)^2,$$

where $y_i \ge 2$ and $\sum y_i \ge 2a$. We obtained that $aP_3 \ge F(\mathbf{y})$ where $\mathbf{y} \in R^b$ is the vector with coordinates formed by the degrees $\{\deg(y) : y \in B\}$. We will see that F is convex in this region, hence all y_i can be replaced with the average of the degrees, i.e., $\sum_{y\in B}\deg(y)/b = e/b$. One obtains

$$aP_3 \ge -ab\frac{e}{b}\Big(\frac{e}{b}-1\Big) + \Big(b\frac{e}{b}\sqrt{\frac{e}{b}-1}\Big)^2.$$

Rearranging one gets (6).

Proof of convexity. Let F_{ij}, F_{ii} denote the partial derivatives, $\mathbf{H}$ the Hessian of F. Then for $i \neq j$ one has

$$F_{ij} = \frac{1}{2} \frac{3y_i - 2}{\sqrt{y_i - 1}} \frac{3y_j - 2}{\sqrt{y_j - 1}},$$

and

$$\begin{aligned} F_{ii} &= -2a + G(\mathbf{y}) \frac{1}{2} \frac{(3y_i - 2)^2}{y_i - 1} + G(\mathbf{y}) \frac{3y_i - 1}{2(y_i - 1)\sqrt{y_i - 1}} \\ &\geq -2a + (G(\mathbf{y}) - 1) \frac{1}{2} \frac{(3y_i - 2)^2}{y_i - 1} + \frac{1}{2} \frac{3y_i - 2}{\sqrt{y_i - 1}} \frac{3y_i - 2}{\sqrt{y_i - 1}}, \end{aligned}$$

where $G(\mathbf{y}) = \sum y_i \sqrt{y_i - 1}$. Since $(3y-2)^2/2(y-1) \geq 6$ for $y > 1$ and $G(\mathbf{y}) \geq \sum y_i = e \geq 2a$ we can write $\mathbf{H}$ as a sum of a positive semidefinite matrices, namely $1/2$ times the tensor product of the vector $\{\frac{3y_i-2}{\sqrt{y_i-1}}\}$ with itself, and a diagonal matrix with diagonal entries exceeding $-2a + 6(G(\mathbf{y}) - 1)$, again a positive definite matrix. Thus $\mathbf{H}$ is positive definite and then F is convex in the region. □

The above theorem is a slightly improved version of a result of Sidorenko [23] which states $P_3 \geq e^3/ab - \Delta e$, where Δ is the maximum degree of G. Concerning general (non-bipartite) graphs, Theorem 1 implies that $P_3 \geq \frac{1}{2} n(d_{\text{ave}})^3 - \frac{3}{2} n \Delta d_{\text{ave}}$. This inequality may also be deduced from a Moore-type bound, established by Alon, Hoory and Linial [2].

4 Graphs without C_6

Theorem 3. *Let $G(A, B)$ be a bipartite graph with e edges and with color classes A and B, $|A| = a$, $|B| = b$. Suppose that G has girth eight. Then for the number edges one has*

$$e \leq (ab)^{2/3} + a + b. \tag{7}$$

Proof. We use induction on the number of vertices if there is any isolated vertex, or a vertex of degree 1. Otherwise, observe, that every pair $x \in A$, $y \in B$ is connected by at most one path of length 3. Thus $P_3 \leq ab$. Comparing this to the lower bound for P_3 in (6) and rearranging we get the Theorem. □

D. de Caen and Székely [7] showed that $e(G) = O((ab)^{2/3})$ assuming $a = O(b^2)$ and $b = O(a^2)$. Later they showed [8] that if G has girth eight and every vertex has degree at least two, then $e \leq 2^{1/3}(ab)^{2/3}$ and here the coefficient $2^{1/3}$ is the best possible by exhibiting a graph with $a = 2s$, $b = s^2$ and $e = 2s^2$. (Note that this does not contradict our result (7) since here $b = e/2$).

Győri [17] observed that in a C_6-free graph G the maximal complete bipartite graphs $K_{\alpha,\beta}$'s with $\alpha, \beta \geq 2$ are edge disjoint (indeed, these are $K_{2,\beta}$'s). Thus one can remove edges from G such that the resulting graph G_0 is C_4-free and $e(G_0) \geq \frac{1}{2}e(G)$. Thus Győri's result combined with Theorem 3 gives that

Corollary 4. *If G is a C_6-free bipartite graph with parts of sizes a and b then $e(G) \leq 2(ab)^{2/3} + 2a + 2b$.* □

More is true. In [14] it was proved that for such a graph

$$e(G) < 2^{1/3}(ab)^{2/3} + 16(a+b) \tag{8}$$

holds. Moreover infinitely many examples show that the coefficient $2^{1/3}$ in the best possible for large a and b with $b = 2a$.

Concerning general (not necessarily bipartite) graphs, it was proved by Bondy and Simonovits [6] in 1974 that a graph on n vertices with at least $100kn^{1+1/k}$ edges contains C_{2k}, a cycle of length $2k$. This was extended into bipartite graphs with parts of sizes of a and b by G. N. Sárközy [22] who showed that such a graph with $\max\{90k(a+b), 20k(ab)^{1+1/k}\}$ edges contains a C_{2k}. Our Corollary 4 gives these for C_6, even a slightly better statement, using the following important reduction theorem.

Lemma 5 (Erdős [9]). *Let G be an arbitrary graph. Then there exists a bipartite subgraph G_0 with $\deg_{G_0}(x) \geq \frac{1}{2}\deg_G(x)$ for all vertices. Especially, $e(G_0) \geq \frac{1}{2}e(G)$.*

Corollary 6. *If G is a C_6-free graph on n vertices then $e(G) \leq 2^{2/3}n^{4/3} + 4n$.* □

It is known that there are C_6-free graphs with at least $(\frac{1}{2} + o(1))n^{4/3}$ edges [19], and the best known lower and upper bounds can be found in [14], (namely $0.533n^{4/3} < \mathbf{ex}(n, C_6) < 0.628n^{4/3}$ for $n > n_0$). Yuansheng and Rowlinson [31] determined $\mathbf{ex}(n, C_6)$ and all extremal graphs for $n \leq 26$.

5 Cube-free graphs

Theorem 7 (Erdős and Simonovits [11]). *Let Q denote the 8-vertex graph formed by the 12 edges of a cube. Then $\mathbf{ex}(n, Q) \leq O(n^{8/5})$.*

The original proof of this is rather complicated. It applies a remarkable regularization process for non-dense bipartite graphs. A somewhat simpler proof was found by Pinchasi and Sharir [21], who were interested in certain geometric incidence problems, and who also extended it to a bipartite version

$$e(G(A,B)) \leq O((ab)^{4/5} + ab^{1/2} + a^{1/2}b). \tag{9}$$

Here we give an even simpler proof which also gives the bipartite version, see (10) below. We only use Theorem 2, Corollary 4 and the power mean inequality (3), but the main ideas are the same as in [11].

Proof of Theorem 7. Let G be an n-vertex Q-free graph. First, applying Erdős' Lemma 5 we choose a large bipartite subgraph $G(A,B)$ of G, $e(G) \le 2e(G(A,B))$.

We say that a hexagon $z_1z_2\dots z_6$ lies *between* the vertices x and y if z_1, z_3, z_5 are neighbors of x and the other vertices of the hexagon are neighbors of y, i.e., $z_1, z_3, z_5 \in N(x)$ and $z_2, z_4, z_6 \in N(y)$ and $\{x, y\} \cap \{z_1, \dots, z_6\} = \emptyset$. The crucial observation is that x and y together with the 6 vertices of a hexagon between them contain a cube Q. So there is no such hexagon in a Q-free graph. Thus we can apply the upper bound for the Turán numbers of C_6, i.e., Theorem 4 and obtain an upper bound for the number of edges uv, $u \in N(x)$, $v \in N(y)$. This gives an upper bound for the number of paths with end vertices x and y. For given x and y we have

$$\begin{aligned}\#xuvy \text{ paths } &= |\{uv \in E(G(A,B)) : u \in N(x) \setminus \{y\}, v \in N(y) \setminus \{x\}\}\\ &\le 2|N(x)|^{2/3}|N(y)|^{2/3} + 2|N(x)| + 2|N(y)|.\end{aligned}$$

Add this up for every $x \in A$, $y \in B$. Let $e := e(G[A,B])$ and use (3) with $(r,s) = (1, 3/2)$. We have

$$\begin{aligned}P_3(G(A,B)) &\le \sum_{x\in A}\sum_{y\in B} 2\deg(x)^{2/3}\deg(y)^{2/3} + 2\deg(x) + 2\deg(y)\\ &= 2\Big(\sum_{x\in A}\deg(x)^{2/3}\Big)\Big(\sum_{y\in B}\deg(y)^{2/3}\Big) + 2be + 2ae\\ &\le 2 \times e^{2/3}a^{1/3} \times e^{2/3}b^{1/3} + 2(a+b)e.\end{aligned}$$

Comparing this to the lower bound in Theorem 2 one obtains that

$$(e-a)(e-b) \le 2e^{1/3}(ab)^{4/3} + 2(a+b)ab.$$

This implies that

$$e \le 2^{3/5}(ab)^{4/5} + 2ab^{1/2} + 2a^{1/2}b. \tag{10}$$

Using $e(G) \le 2e$ we obtain

$$e(G) \le n^{8/5} + (2n)^{3/2} \tag{11}$$

finishing the proof. □

If we use (8) instead of Corollary 4 then the above calculation gives

Theorem 8.

$$\mathbf{ex}(a,b,Q) \le 2^{1/5}(ab)^{4/5} + 9(ab^{1/2} + a^{1/2}b) \tag{12}$$

and

$$\mathbf{ex}(n,Q) \le 2^{-2/5}n^{8/5} + 13n^{3/2}. \tag{13}$$

6 A lower bound on the number of C_4's

Let $N(G, H)$ denote the number of subgraphs of G isomorphic to H. This function is even more important than the original Turán problem. Here we consider only one of the simplest cases, $H = C_4$.

It was observed and used many times that for sufficiently large e the graph G contains at least $\Omega(e^4/n^4)$ copies of C_4. This result goes back to Erdős (1962) and was published, e.g., in Erdős and Simonovits [11] in an asymptotic form ($N(G, C_4) > (1/3)e^4/n^4$ for $n > Cn^{3/2}$). The following simple form has the advantage that it is valid for arbitrary n and e.

Lemma 9 (see [15]). *Let G be a graph with e edges and n vertices. Then*

$$N(G, C_4) \geq 2\frac{e^3(e-n)}{n^4} - \frac{e^2}{2n} \geq 2\frac{e^4}{n^4} - \frac{3}{4}en. \tag{14}$$

Allen, Keevash, Sudakov, and Verstraëte [1] gave a bipartite version of Lemma 9. Here we state their result in a slightly stronger form (it is valid for all values of a, b and e). Note that the formula is not symmetric in A and B.

Lemma 10. *Let G be a bipartite graph with parts A and B of sizes a and b and e edges. Then the number of 4-cycles in G is at least*

$$\frac{e^2(e-b)^2 - e(e-b)ba(a-1)}{4b^2a(a-1)}. \tag{15}$$

For completeness we present the proofs of the above Lemmas (below and in the Appendix). But we will need a slightly stronger and more technical version.

Lemma 11. *Let G be a bipartite graph with parts A and B of sizes a and b and e edges. Let $D(x)$ denote $\sum_{y\in N(x)}(\deg(y)-1)$. Then the number of 4-cycles in G is at least*

$$\frac{1}{4(a-1)}\Big(\sum_{x\in A} D(x)^2\Big) - \frac{1}{4}\Big(\sum_{x\in A} D(x)\Big). \tag{16}$$

Proof. We have

$$\begin{aligned} N(G, C_4) &= \sum_{\{x,x'\}\subset A} \binom{d(x,x')}{2} = \frac{1}{2}\sum_{x\in A}\left(\sum_{x'\in A\setminus x}\binom{d(x,x')}{2}\right) \\ &= \frac{1}{2}\sum_{x\in A}(a-1)\binom{\sum_{x'\in A\setminus x} d(x,x')/(a-1)}{2} \\ &= \frac{a-1}{2}\sum_{x\in A}\binom{D(x)/(a-1)}{2} \\ &= \frac{1}{4(a-1)}\left(\sum_{x\in A} D(x)^2\right) - \frac{1}{4}\left(\sum_{x\in A} D(x)\right). \end{aligned}$$

□

Note that Lemma 11 easily implies Lemma 10. Indeed, observe that for $e(e-b) < ba(a-1)$ the right-hand side of (15) is negative, so we may suppose that $(e^2/b) - e \geq a(a-1)$. Use Chauchy–Schwartz for $\sum_{x\in A} D(x)$. We obtain

$$\sum_{x\in A} D(x) = \sum_{x\in A}\Big(\sum_{y\in B, xy\in E(G)} (\deg(y)-1)\Big) = \sum_{y\in B}\deg(y)^2 - \sum_{y\in B}\deg(y) \geq \frac{e^2}{b} - e.$$

Use Chauchy–Schwartz again for $\sum D(x)^2$. We have

$$\sum_{x\in A} D(x)^2 \geq \frac{1}{a}\Big(\sum_{x\in A} D(x)\Big)^2.$$

Now Lemma 11 gives that $N(G, C_4) \geq (N^2/4a(a-1)) - (N/4)$ for $N := \sum_{x\in A} D(x)$. Since $N \geq (e^2/b) - e \geq a(a-1)$ the polynomial $p(N) := N^2/a(a-1) - N$ is increasing and we get $N(G, C_4) \geq p(N) \geq p(e(e-b)/b))$. □

7 Cubes with a diagonal

Theorem 12 (Erdős and Simonovits [11]). *Let Q^+ denote the* 8*-vertex graph formed by the* 12 *edges of a cube with a long diagonal. Then* $\mathbf{ex}(n, Q^+) \leq O(n^{8/5})$.

Here we give a simpler proof which also gives a stronger bipartite version.

Theorem 13. $\mathbf{ex}(a, b, Q^+) \leq 2^{3/5}(ab)^{4/5} + O(ab^{1/2} + a^{1/2}b)$.

Using again Erdős' Lemma 5 and $a + b = n$ we get

$$\mathbf{ex}(n, Q^+) \leq n^{8/5} + O(n^{3/2}). \tag{17}$$

Proof of Theorem 13. Let G be an n-vertex Q^+-free bipartite graph with classes A and B. The main idea is the same as in [11] and in the proof of Theorem 7. The crucial observation is that an edge $xy \in E(G)$ together with the 6 vertices of a hexagon between them form a Q^+. So there is no such hexagon in a Q^+-free graph between the neighborhoods of two connected vertices. Thus we can apply the upper bound for the Turán numbers of C_6, i.e., Theorem 4 and obtain an upper bound for the number of edges $x'y'$, $y' \in N(x)$, $x' \in N(y)$ for $xy \in E(G)$. This gives an upper bound for the number of four cycles containing the edge xy.

$$\begin{aligned}\#xy'x'y \text{ four cycles } &= |\{x'y' \in E(G(A,B)) : y' \in N(x)\setminus\{y\}, x' \in N(y)\setminus\{x\}\}\\ &\leq 2(|N(x)|-1)^{2/3}(|N(y)|-1)^{2/3} + 2|N(x)| + 2|N(y)| - 2.\end{aligned}$$

Add this up for every $x \in A$, $y \in B$, $xy \in E(G)$ and apply (3) for $\sum_{xy\in E}(\deg(y) - 1)^{2/3}$ with $(r, s) = (1, 3/2)$ for every x. We obtain

$$4N(G, C_4) \le \sum_{x\in A} \sum_{y\in N(x)} 2(\deg(x) - 1)^{2/3}(\deg(y) - 1)^{2/3} + 2\deg(x) + 2\deg(y) - 2$$
$$= 2\Big(\sum_{x\in A}(\deg(x) - 1)^{2/3} \deg(x)^{1/3} D(x)^{2/3}\Big) + 2\Big(\sum_{y\in B} D(y)\Big) + 2\Big(\sum_{x\in A} D(x)\Big).$$

Apply Hölder inequality with $1/p = 2/3$ and $1/q = 1/3$ in the first term. We obtain it is at most

$$\le 2\Big(\sum_{x\in A}(\deg(x) - 1)\deg(x)^{1/2}\Big)^{2/3}\Big(\sum_{x\in A} D(x)^2\Big)^{1/3}. \tag{18}$$

From now on, to save time and energy, and to better emphasize the main steps of calculation we only sketch the proof. Compare the obvious leading terms in the lower and upper bounds (16) and (18) for $N(G, C_4)$, we have

$$\frac{1}{a-1}\Big(\sum_{x\in A} D(x)^2\Big) \ll 4N(G, C_4) \ll 2\Big(\sum_{x\in A}(\deg(x) - 1)\deg(x)^{1/2}\Big)^{2/3}\Big(\sum_{x\in A} D(x)^2\Big)^{1/3}$$

yielding

$$\Big(\sum_{x\in A} D(x)^2\Big) \ll (2(a-1))^{3/2}\Big(\sum_{x\in A}(\deg(x) - 1)\deg(x)^{1/2}\Big). \tag{19}$$

On the left-hand side we can use Cauchy–Schwartz and on the right-hand side we apply (3) with $(r, s) = (3/2, 2)$. We obtain

$$\frac{1}{a}\Big(\sum_{x\in A} D(x)\Big)^2 \ll (2(a-1))^{3/2} a^{1/4}\Big(\sum_{x\in A}(\deg(x) - 1)d(x)\Big)^{3/4}.$$

Rearranging we have

$$\Big(\sum_{x\in A} D(x)\Big)^2 \ll 2^{3/2} a^{11/4}\Big(\sum_{y\in B} D(y)\Big)^{3/4}. \tag{20}$$

Exchange the role of A and B, we get

$$\Big(\sum_{y\in B} D(y)\Big)^2 \ll 2^{3/2} b^{11/4}\Big(\sum_{x\in A} D(x)\Big)^{3/4}.$$

Multiply the above two inequalities, take 4^{th} power, we get

$$2^{12}(ab)^{11} \gg \Big(\sum_{x\in A} D(x)\Big)^5\Big(\sum_{y\in B} D(y)\Big)^5 \ge \Big(\frac{e^2}{a} - e\Big)^5\Big(\frac{e^2}{b} - e\Big)^5$$

leading to $2^{12}(ab)^{16} \gg e^{20}$. □

8 Appendix

(1) *A direct proof of the Mulholland–Smith inequality* (5) *concerning the number of 3-walks using only high school algebra.*

Considering the middle edge of the 3-walks one obtains that

$$W_3 = \sum_{x\in V}\sum_{y\in N(x)} \deg(x)\deg(y).$$

Here we have $2e = nd_{\text{ave}}$ terms. Our aim is to separate the variables in the products $\deg(x)\deg(y)$ so next we use first that the $2e$-dimensional quadratic inequality (quadratic mean is greater than or equal the arithmetic mean), second we use (for 2 variables) that the arithmetic mean is greater than or equal the harmonic mean, then third time we use again (this time for $2e$ variables) that arithmetic ≥ harmonic. One obtains that

$$\begin{aligned}\sqrt{\frac{W_3}{nd_{\text{ave}}}} &= \sqrt{\frac{\sum_{x\in V}\sum_{y\in N(x)}\deg(x)\deg(y)}{2e}}\\ &\geq \frac{\sum_{x\in V}\sum_{y\in N(x)}\sqrt{\deg(x)\deg(y)}}{2e}\\ &\geq \frac{1}{2e}\Bigg(\sum_{x\in V}\sum_{y\in N(x)}\frac{2}{\frac{1}{\deg(x)}+\frac{1}{\deg(y)}}\Bigg)\\ &\geq 2e\Bigg(\sum_{x\in V}\sum_{y\in N(x)}\frac{\frac{1}{\deg(x)}+\frac{1}{\deg(y)}}{2}\Bigg)^{-1} = \frac{2e}{\sum_{x\in V}1} = \frac{2e}{n} = d_{\text{ave}}.\end{aligned}$$

□

(2) *Proof of Lemma 9 concerning the number of C_4's.*

Denote the number of x, y-paths of length two by $d(x, y)$. We have

$$\overline{\overline{d}} := \binom{n}{2}^{-1}\sum_{x,y\in V(G)} d(x,y) = \binom{n}{2}^{-1}\sum_{x\in V(G)}\binom{\deg(x)}{2} \geq \binom{n}{2}^{-1} n\binom{2e/n}{2}. \tag{21}$$

Therefore, $\overline{\overline{d}} \geq \frac{2e(2e-n)}{n^2(n-1)}$. Moreover

$$N(G, C_4) = \frac{1}{2}\sum_{x,y\in V(G)}\binom{d(x,y)}{2} \geq \frac{1}{2}\binom{n}{2}\binom{\overline{\overline{d}}}{2}. \tag{22}$$

We may suppose that the middle term in (14) is positive, which implies that $\frac{2e(2e-n)}{n^2(n-1)} \geq 1/2$. The paraboloid $\binom{x}{2}$ is increasing for $x \geq 1/2$. So we may substitute the lower bound of $\overline{\overline{d}}$ from (21) into (22) and a little algebra gives (14). □

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